ON A NEW TRANSFORMATION SCHEME RELATED TO A 4-DIMENSIONAL LORENTZ TRANSFORMATION*

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ABSTRACT. In a four-space R_4 a new real transformation scheme has been so framed that a new representation of a four-dimensional Lorentz transformation is obtained showing the connection between the respective transformation coefficients. The method is a modification of that used in an earlier paper where the representation of a five-dimensional Lorentz transformation was considered. It is to be noted that under this new transformation scheme one can set up a system of 'Dirac Equations'.

INTRODUCTION

In a recent paper (Ghosh 1964) a new representation of a five dimensional Lorentz transformation was obtained by utilizing the real transformation coefficients of a new transformation scheme defined in a four-space R_4 with coordinates x^0, x^1, x^2, x^3 which leaves invariant an elementary antisymmetric tensor C_{pq} with 4 non-vanishing components $C_{01} = -C_{10} = 1$, $C_{23} = -C_{32} = 1$. The object of the present paper is to modify the transformation scheme so that the representation corresponds to a four-dimensional Lorentz transformation both proper and improper. Further, it is shown how under this transformation scheme one can set up a system of Dirac Equations derived from an invariant divergence equation.

1. Starting with the set of general transformation equations in R_4 from coordinates x^r to x'^r and vice versa

$$x'^r = \phi^r(x^0, x^1, x^2, x^3),$$

$$x^r = \phi^{-1r}(x'^0, x'^1, x'^2, x'^3), \quad (r = 0, 1, 2, 3) \qquad \dots \quad (1.1)$$

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with 16 covariant transformation coefficients $\partial x^r/\partial x'^p$ we first impose the following 8 restrictive conditions on the function ϕ' given by

$$
\begin{aligned}
\binom{0}{0} &= \lambda\binom{3}{3}, & \binom{0}{3} &= -\lambda\binom{3}{0},\\
\binom{0}{1} &= \lambda\binom{3}{2}, & \binom{0}{2} &= \lambda\binom{3}{1},\\
\binom{1}{0} &= \lambda\binom{2}{3}, & \binom{2}{0} &= \lambda\binom{1}{3},\\
\binom{1}{1} &= \lambda\binom{2}{2}, & \binom{2}{1} &= -\lambda\binom{1}{2},
\end{aligned}
\qquad \ldots \quad (1.2)
$$

where λ is either $+1$ or -1, the symbol $\binom{r}{p}$ denoting $\partial x'^r/\partial x'^p$. If $\lambda = 1$, the system of transformations will possess group property and we call it a proper general transformation. With $\lambda = -1$, the system does not possess group property and we call it an improper general transformation.

The conditions (1.2) admit of being expressed in terms of contravariant transformation coefficients. These are of the same type as (1.2) provided the symbol $\binom{r}{p}$ is replaced by $\partial x'^r/\partial x^p$.

2. Let us now define a special transformation in the above scheme by introducing an antisymmetric covariant tensor C'_{pq} with non-vanishing components $C'_{01} = -C'_{10} = 1$, $C'_{23} = -C'_{32} = 1$, and postulating that it remains invariant. Referring to (1.1) the transformation equation of C'_{pq} is given by

$$C''_{rs} = C'_{pq}\frac{\partial x^p}{\partial x'^r}\cdot\frac{\partial x^q}{\partial x'^s} \quad (r, s = 0, 1, 2, 3) \qquad \ldots \quad (2.1)$$

For invariance of C'_{pq} the full set of conditions are given in my last paper (Ghosh, 1964). Here, in addition to conditions (1.2) two more conditions independent of λ have to be satisfied. These are

$$
\begin{aligned}
\binom{01}{01} + \binom{01}{23} &= 1,\\
\binom{02}{01} + \binom{02}{23} &= 0,
\end{aligned}
\qquad \ldots \quad (2.2)
$$

where the symbol $\binom{pq}{rs}$ denotes $\binom{p}{r}\binom{q}{s} - \binom{p}{s}\binom{q}{r}$.

Alternatively we can write (2.2) as

$$
\begin{aligned}
\binom{01}{01} + \binom{23}{01} &= 1,\\
\binom{01}{02} + \binom{23}{02} &= 0.
\end{aligned}
\qquad \ldots \quad (2.3)
$$

This special transformation will be called unimodular tensor transformation, proper if $\lambda = 1$ and improper if $\lambda = -1$. The contravariant tensor associated to C_{pq} will be denoted by C^{pq} having non-vanishing components $C^{01} = -C^{10} = 1$, $C^{23} = -C^{32} = 1$. It may be noted that the contravariant and covariant coefficients of a unimodular tensor transformation are connected by the equation

$$\frac{\partial x'^q}{\partial x^s} = C_{rs} C^{pq} \frac{\partial x^r}{\partial x'^p} . \qquad \dots \quad (2.4)$$

This gives

$$\begin{aligned}
&\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{Bmatrix} 2 \\ 0 \end{Bmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{Bmatrix} 3 \\ 0 \end{Bmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \\
&\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cdot \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cdot \\
&\begin{Bmatrix} 0 \\ 2 \end{Bmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \\
&\begin{Bmatrix} 0 \\ 3 \end{Bmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot
\end{aligned} \qquad \dots \quad (2.5)$$

where we have used the notation $\begin{Bmatrix} p \\ q \end{Bmatrix}$ to denote $\partial x'^p / \partial x^q$.

The tensors C^{pq}, C_{pq} may be regarded as metric tensors in R_4. Raising and lowering of indices may be performed according to the rule

$$A_q C^{pq} = A^p, \quad A^p C_{pq} = A_q. \qquad \dots \quad (2.6)$$

so that $\qquad C^{pq} C_{pr} = C^{qp} C_{rp} = \delta_r^q.$

Thus $\qquad A^0 = A_1,\ A^1 = -A_0,\ A^2 = A_3,\ A^3 = -A_2.$

We note here the relations

$$A_p A^p = 0 \qquad A_p B^p + B_p A^p = 0. \qquad \dots \quad (3.7)$$

Consider now a mixed tensor $M_r{}^p$ defined in terms of 4 quantities h_k, forming a vector, by means of an equation

$$M_r{}^p = T^{kp}{}_r h_k \ (k = 0, 1, 2, 3), \qquad \dots \quad (3.8)$$

where T's are connecting tensors having the following structure:

$$T^{0}{}_r{}^p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad T^{1}{}_r{}^p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$T^{2}{}_r{}^p = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T^{3}{}_r{}^p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (3.9)$$

Thus

$$M_0{}^0 = M_1{}^1 = -h_2, \quad M_2{}^2 = M_3{}^3 = h_2, \quad M_2{}^1 = -M_0{}^3 = -h_1,$$

$$M_3{}^0 = -M_1{}^2 = h_1, \quad M_2{}^0 = M_1{}^3 = h_0+h_3 \quad M_0{}^2 = M_3{}^1 = -h_0+h_3,$$

$$M_1{}^0 = M_0{}^1 = M_3{}^2 = M_2{}^3 = 0. \quad \dots \quad (3.10)$$

Applying the rule (2.6) one can see that $M^p{}_r$ will have the above structure if it is taken in a bilinear form $A^pB_r - B^pA_r$, A^p, B_r being arbitrary tensors of rank 1.

Inverting (3.8) we write

$$h_k = \frac{1}{4} T_{kp}^{\ \ r} M_r^p , \quad \dots \quad (3.11)$$

where

$$T_{kp}^{\ \ r} = g_{kl} T^{lr}_{\ \ p} , \quad \dots \quad (3.12)$$

g_{kl} denoting the tensor with non-vanishing components

$$g_{00} = -1, \; g_{11} = g_{22} = g_{33} = 1.$$

Let $M_r{}^p$ undergo a unimodular tensor transformation

$$M'^p_r = M^q_s \left\{ {p \atop q} \right\} \left({s \atop r} \right) , \quad \dots \quad (3.13)$$

where the transformation coefficients satisfy relations (1.2), (2.2,4). It is found that after the transformation $M'_r{}^p$ will have the same structure as that of $M_r{}^p$, so that we can express $M'_r{}^p$ in the same way as (3.8)

that is, $$M'^p_r = T^{kp}{}_r \, h'_k , \quad \dots \quad (3.14)$$

where h'_k denotes the transformed vector h_k induced by the unimodular tensor transformation with regard to $M_r{}^p$

Inverting (3.14) we write

$$h'_k = \frac{1}{4} T^r_{kp} M'^p_r . \quad \dots \quad (3.15)$$

Using (3.13), the above becomes

$$h'_k = \frac{1}{4} T^r_{kp} M^q_s \left\{ {p \atop q} \right\} \left({s \atop r} \right) \cdot \quad \dots \quad (3.16)$$

Writing $h'_k = a^l{}_k h_l$ and $M^q_s = T^{lq}_s h_l$, we obtain

$$a^l{}_k = \frac{1}{4} T^r{}_{kp} T^{lq}_s \left\{ {p \atop q} \right\} \left({s \atop r} \right). \quad \dots \quad (3.17)$$

Observing that the invariant $M_r^p M_p^r = 4(h_1^2+h_2^2+h_3^2-h_0^2)$ we remark that when M_r^p undergoes a unimodular tensor transformation the vector h_k undergoes a Lorentz transformation leaving $h_1^2+h_2^2+h_3^2-h_0^2$ invariant.

4. In matrix form (3.17) is expressible as

$$\begin{bmatrix} a_0^0 & a_1^0 & a_2^0 & a_3^0 \\ a_0^1 & a_1^1 & a_2^1 & a_3^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \begin{pmatrix} 03, 12 \\ 03, 12 \end{pmatrix} & \begin{pmatrix} 03, 12 \\ 13, 02 \end{pmatrix} & \begin{pmatrix} 03, 12 \\ 01, 32 \end{pmatrix} & \begin{pmatrix} 03, 12 \\ 12, 30 \end{pmatrix} \\ \begin{pmatrix} 13, 02 \\ 03, 12 \end{pmatrix} & \begin{pmatrix} 13, 02 \\ 13, 02 \end{pmatrix} & \begin{pmatrix} 13, 02 \\ 01, 32 \end{pmatrix} & \begin{pmatrix} 13, 02 \\ 12, 30 \end{pmatrix} \\ \begin{pmatrix} 01, 32 \\ 03, 12 \end{pmatrix} & \begin{pmatrix} 01, 32 \\ 13, 02 \end{pmatrix} & \begin{pmatrix} 01, 32 \\ 01, 32 \end{pmatrix} & \begin{pmatrix} 01, 32 \\ 12, 30 \end{pmatrix} \\ \begin{pmatrix} 12, 30 \\ 03, 12 \end{pmatrix} & \begin{pmatrix} 12, 30 \\ 13, 02 \end{pmatrix} & \begin{pmatrix} 12, 30 \\ 01, 32 \end{pmatrix} & \begin{pmatrix} 12, 30 \\ 12, 30 \end{pmatrix} \end{bmatrix} \qquad \ldots \quad (4.1)$$

where the symbol $\begin{pmatrix} pq, rs \\ tu, vw \end{pmatrix}$ denotes $\begin{pmatrix} pq \\ tu \end{pmatrix} + \begin{pmatrix} pq \\ vw \end{pmatrix} + \begin{pmatrix} rs \\ tu \end{pmatrix} + \begin{pmatrix} rs \\ vw \end{pmatrix}$.

By elementary operations utilising the relations (1.2) the determinant of the right-hand matrix in (4.1) may be reduced to

$$4 \times \begin{vmatrix} \frac{1}{2}\begin{pmatrix} 12 \\ 12 \end{pmatrix} & \begin{pmatrix} 12 \\ 02 \end{pmatrix} & \begin{pmatrix} 12 \\ 01 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 12 \\ 03 \end{pmatrix} \\ \begin{pmatrix} 02 \\ 12 \end{pmatrix} & \begin{pmatrix} 02 \\ 02 \end{pmatrix} + \begin{pmatrix} 02 \\ 13 \end{pmatrix} & \begin{pmatrix} 02 \\ 01 \end{pmatrix} + \begin{pmatrix} 02 \\ 32 \end{pmatrix} & \begin{pmatrix} 02 \\ 03 \end{pmatrix} \\ \begin{pmatrix} 01 \\ 12 \end{pmatrix} & \begin{pmatrix} 01 \\ 02 \end{pmatrix} + \begin{pmatrix} 32 \\ 02 \end{pmatrix} & \begin{pmatrix} 01 \\ 01 \end{pmatrix} + \begin{pmatrix} 01 \\ 32 \end{pmatrix} & \begin{pmatrix} 01 \\ 03 \end{pmatrix} \\ \frac{1}{2}\begin{pmatrix} 03 \\ 12 \end{pmatrix} & \begin{pmatrix} 03 \\ 02 \end{pmatrix} & \begin{pmatrix} 03 \\ 01 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 03 \\ 03 \end{pmatrix} \end{vmatrix} \qquad \ldots \quad (4.2)$$

Introducing the quantities

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$a_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \qquad b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$a_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \qquad b_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$a_4 = \binom{0}{1}\binom{1}{0} + \binom{0}{2}\binom{1}{3}. \qquad b_4 = \binom{0}{1}\binom{1}{3} - \binom{0}{2}\binom{1}{0}.$$

$$a_5 = \binom{0}{1}\binom{1}{1} + \binom{0}{2}\binom{1}{2}. \qquad b_5 = \binom{0}{1}\binom{1}{2} - \binom{0}{2}\binom{1}{1}.$$

$$a_6 = \binom{1}{0}\binom{1}{1} + \binom{1}{3}\binom{1}{2}. \qquad b_6 = \binom{1}{0}\binom{1}{2} - \binom{1}{3}\binom{1}{1}.$$

$$c_1 = \frac{1}{2}\binom{0}{0}^2 + \frac{1}{2}\binom{0}{3}^2, \quad c_2 = \frac{1}{2}\binom{0}{1}^2 + \frac{1}{2}\binom{0}{2}^2,$$

$$c_3 = \frac{1}{2}\binom{1}{0}^2 + \frac{1}{2}\binom{1}{3}^2, \quad c_4 = \frac{1}{2}\binom{1}{1}^2 + \frac{1}{2}\binom{1}{2}^2 \tag{4.3}$$

(4.2) may be expressed as

$$4 \times \begin{vmatrix} \lambda c_4 & \lambda a_6 & -\lambda b_6 & \lambda c_3 \\ \lambda a_5 & \lambda(a_1+a_4) & \lambda(b_4-b_1) & \lambda a_3 \\ b_5 & b_1+b_4 & a_1-a_4 & b_3 \\ \lambda c_2 & \lambda a_2 & -\lambda b_2 & \lambda c \end{vmatrix} \tag{4.4}$$

Omitting the multiplier 4, the Laplace expansion of (4.4) gives

$$\begin{aligned} &\lambda^2(c_1a_3-c_3a_5)(b_4b_2+b_1b_2+a_1a_2-a_2a_4)(-\lambda) \\ &-\lambda(c_4b_3-c_3b_5)(b_2a_1+b_2a_4+a_2b_4-a_2b_1)(-\lambda^2) \\ &+\lambda^2(c_1c_4-c_2c_3)(a_1^2-a_4^2-b_4^2-b_1^2)\lambda \\ &+\lambda(a_5b_3-b_5a_3)(a_6b_2-b_6a_2)(-\lambda^2) \\ &-\lambda^2(c_1a_5-c_2a_3)(a_6a_1-a_6a_4+b_6b_4+b_1b_6)\lambda \\ &+\lambda(c_1b_5-c_2b_3)(a_6b_4-a_6b_1+b_6a_1+b_6a_4)\lambda^2. \end{aligned} \tag{4.5}$$

Let us now take note of the following identities :

$$\begin{aligned} a_1a_2+b_1b_2 &= 2c_1a_5, & a_1b_2-b_1a_2 &= -2c_1b_5, \\ a_2a_4-b_2b_4 &= 2c_2a_3, & a_2b_4+b_2a_4 &= 2c_2b_3, \\ a_4a_6-b_4b_6 &= 2c_3a_5, & a_4b_6+b_4a_6 &= 2c_3b_5. \\ a_1a_6+b_1b_6 &= 2c_4a_3, & a_1b_6-b_1a_6 &= -2c_4b_3. \end{aligned} \tag{4.6}$$

$$a_5b_3-b_5a_3 = a_6b_2-a_2b_6.$$

Substituting from the above (4.5) becomes

$$\lambda^3[4c_2c_4a_3^2+4c_1c_3a_5^2+4c_2c_4b_3^2+4c_1c_3b_5^2$$
$$-4(c_1c_4+c_2c_3)(a_3a_5+b_3b_5)-(a_5b_3-b_5a_3)^2+(c_1c_4-c_2c_3)(a_1^2-a_4^2-b_4^2+b_1^2)]. \quad (4.7)$$

Using further the typical relations

$$4c_1c_4 = a_1^2+b_1^2,\ 4c_1c_2 = a_2^2+b_2^2,\ 4c_1c_3 = a_3^2+b_3^2,$$
$$4c_2c_3 = a_4^2+b_4^2,\ 4c_2c_4 = a_5^2+b_5^2,\ 4c_3c_4 = a_6^2+b_6^2. \quad \dots \quad (4.8)$$

(4.7) can be expressed as

$$\lambda^3[2(c_1c_4+c_2c_3)-(a_3a_5+b_3b_5)]^2. \quad \dots \quad (4.9)$$

By a straight-forward calculation the above can be shown to be

$$\lambda^3 \left[\frac{1}{2}\left\{\binom{0}{0}\binom{1}{1} - \binom{0}{1}\binom{1}{0} + \binom{0}{2}\binom{1}{3} - \binom{0}{3}\binom{1}{2}\right\}^2\right]^2. \quad (4.10)$$

Referring to (2.2), the value of the determinant (4.2) is therefore λ^3, which is 1 if $\lambda = 1$ and -1 if $\lambda = -1$. Thus the representation (4.1) corresponds to 4-dimensional Lorentz transformation both proper and improper. The relevant conditions to be satisfied by the transformation coefficients $\binom{p}{q}$, being given by (1.2), (2.2) The linearity with regard to the Lorentz transformation will follow if we start with the set of transformation equations (1.1) all linear, so that the coefficients of transformation are constants.

5. We shall now show that under this new transformation scheme one can set up a system of 'Dirac equations'. Let us first construct the differential operator Δ^p_q

given by
$$\Delta^p_q = T^{kp}_{\ q}\partial_k. \quad \dots \quad (5.1)$$

where ∂_k denotes $\partial/\partial x^k$. The four components of the operator Δ^p_q are given by

$$\Delta_0^{\ 0} = \Delta_1^{\ 1} = -\Delta_2^{\ 2} = -\Delta_3^{\ 3} = -\partial_2,\ \Delta_2^{\ 1} = -\Delta_0^{\ 3} = \Delta_1^{\ 2} = -\Delta_3^{\ 0} = -\partial_1.$$
$$\dots \quad (5.2)$$
$$\Delta_2^{\ 0} = \Delta_1^{\ 3} = \partial_0+\partial_3,\ \ \Delta_0^{\ 2} = \Delta_3^{\ 1} = -\partial_0+\partial_3,\ \ \Delta_1^{\ 0} = \Delta_0^{\ 1} = \Delta_3^{\ 2} = \Delta_2^{\ 3} = 0.$$

Consider now the invariant divergence equation

$$\Delta^p_q M^q_p = 0, \quad \dots \quad (5.3)$$

where
$$M^q_p = A^qB_p - B^qA_p.$$

Expanding (5.3) we get

$$(\Delta^p_q A^q)B_p + A^q(\Delta^p_q B_p) - (\Delta^p_q B^q)A_p - B^q(\Delta^p_q A_p) = 0. \quad \dots \quad (5.4)$$

It is easy to show that

$$A^q(\Delta^p_q B_p) = -(\Delta^p_q B^q)A_p,$$

where A and B can be interchanged.

Hence (5.4) becomes

$$(\Delta^p_q A^q)B_p - (\Delta^p_q B^q)A_p = 0, \quad \ldots \quad (5.5)$$

Introducing the subsidiary equations

$$\Delta^p_q A^q = mB^p,$$

$$\Delta^p_q B^q = nA^p, \quad (m, n \text{ arbitrary constants}) \quad \ldots \quad (5.6)$$

into (5.5), it is automatically satisfied by virtue of the first of the relations (3.7) The symmetry in (5 4) shows that m and n are equal.

The equation (5.5) is also satisfied if we introduce the subsidiary equations

$$\Delta^p_q A^q = mA^p,$$

$$\Delta^p_q B^q = -mB^p. \quad \ldots \quad (5.7)$$

and take into account the 2nd of the relations (3.7) Combining the two sets of equations in (5.6) with $m = n$, we obtain the wave equation

$$\Delta^r_p \Delta^p_q A^q = m^2 A^r. \quad \ldots \quad (5.8)$$

6. Let us now consider the infinitesimal unimodular tensor transformation. Starting with the infinitesimal transformation equations

$$x'^r = x^r + \epsilon f^r(x^0, x^1, x^2, x^3). \quad \ldots \quad (6.1)$$

where ϵ is an infinitely small quantity we impose the conditions (1.2) with $\lambda = 1$ to (6.1) and obtain

$$\begin{aligned} f_0^0 &= -f_3^3, & f_3^0 &= -f_0^3, \\ f_1^0 &= -f_2^3, & f_2^0 &= -f_1^3, \\ f_0^1 &= f_3^2, & f_0^2 &= -f_3^1, \\ f_1^1 &= f_2^2, & f_1^2 &= -f_2^1, \end{aligned} \quad \ldots \quad (6.2)$$

where f^r_p denotes $\partial f^r/\partial x^p$. Further, the imposition of conditions (2.2) yields

$$f_0^0 + f_1^1 = 0, \quad f_2^1 - f_0^3 = 0. \quad \ldots \quad (6.3)$$

An infinitesimal unimodular tensor transformation will be defined by (6.1) where f's are restricted by (6.2, 3). Its connection with infinitesimal Lorentz transformation can be exhibited by means of the formula

$$\omega_k{}^l = \frac{\epsilon}{4}\, T_{kp}{}^r [T_r{}^{lm} f_m{}^p - T_m{}^{lp} f_r{}^m], \quad \ldots \quad (6.4)$$

where ω_k^l denotes the coefficient of infinitesimal Lorentz transformation. Evaluating the above we get

$$\omega_0^0 = \omega_1^1 = \omega_2^2 = \omega_3^3 = 0$$

$$
\begin{aligned}
\omega_1^0 &= \omega_0^1 = -\epsilon(f_0^1 + f_1^0), & \omega_2^0 &= \omega_0^2 = \epsilon(f_0^2 + f_2^0), \\
\omega_3^0 &= \omega_0^3 = 2\epsilon f_0^0, & \omega_2^1 &= -\omega_1^2 = 2\epsilon f_2^1, \qquad \ldots \quad (6.5) \\
\omega_3^1 &= -\omega_1^3 = \epsilon(f_0^1 - f_1^0), & \omega_3^2 &= -\omega_2^3 = -\epsilon(f_0^2 - f_2^0).
\end{aligned}
$$

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